

ASYMPTOTIC BEHAVIOR OF OPERATOR SEQUENCES ON KB-SPACES

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ABSTRACT. The concept of a constrictor was used by several mathematicians to characterize the asymptotic behavior of operators. In this paper we show that a positive LR-sequence on KB-spaces is mean ergodic if the LR-sequence has a weakly compact attractor. Moreover if the weakly compact attractor is an order interval, then a Markovian LR-sequence converges strongly to the finite dimensional fixed space. As a consequence we investigate also stability of LR-sequences of positive operators and existence of lower bound functions on KB-spaces.

1. INTRODUCTION

In [3], it is proved that if T is Markov operator on a KB-space then T is mean ergodic and satisfies $\dim(\text{Fix}(T)) < +\infty$ whenever there exist a function $g \in E_+$ and a real $0 \leq \eta < 1$ such that $\lim_{n \rightarrow \infty} \text{dist}(\frac{1}{n} \sum_{k=0}^{n-1} T^k x, [-g, g] + \eta B_E) = 0$ for every $x \in E_+ \cap U_E$. In this paper, we extend this result to any Markov LR-sequence on KB-spaces.

Let X be Banach space and $I_X = I$ be the identity operator. Let E be a Banach lattice. Then $E_+ := \{x \in E : x \geq 0\}$ denotes the positive cone of E . On $\mathcal{L}(E)$ there is a canonical order given by $S \leq T$ if $Sx \leq Tx$ for all $x \in E_+$. If $0 \leq T$, then T is called positive. The dual space E' equipped with the canonical order is again a Banach lattice. Instead of the operations \sup and \inf on E we often write \vee and \wedge , respectively. For $x \in E_+$ we denote by $[-x, x] := \{y \in E : |y| \leq x\}$ the order interval generated by x . A linear subspace of E is an ideal if $[-|x|, |x|] \subseteq I$ for all $x \in I$. An ideal I in E is called a band if for every subset $M \subseteq I$ such that $\sup M$ exists in E one has $\sup M \in I$. A band I in E is called a projection band if there is a linear projection $P : E \rightarrow I$ such that $0 \leq Px \leq x$ for all $x \in E_+$. Such a projection is called a band projection. A Banach lattice E is called a KB-space whenever every increasing norm bounded sequence of E_+ is norm convergent. In particular, it follows that every KB-space has order continuous norm. All reflexive Banach lattice and AL-space are examples of KB-spaces. The following theorem is a combination of results by many mathematicians, for proofs see [1, 15, 18].

Theorem 1.1. *For a Banach lattice E the following statements are equivalent:*

- E is a KB-space.
- E is a band of E'' .

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- E is weakly sequentially complete.
- c_0 is not embeddable in E .
- c_0 is not lattice embeddable in E .

2. LR-SEQUENCES

The main tool in this section is the operator sequence on Banach space X . A family $\Theta = (T_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}(X)$ is called an operator sequence. The sequence Θ is strongly convergent if the norm-limit $\|\cdot\| - \lim_{n \rightarrow \infty} T_n x$ exists for each $x \in X$. A vector x is called a fixed vector for the sequence Θ if $T_n x = x$ for each $n \in \mathbb{N}$. We denote by $\text{Fix}(\Theta)$ the set of all fixed vectors of Θ .

The following important concept was introduced by H.P.Lotz [14] and F. Rübiger [17], we use the modified terminology from [6] called LR-sequences.

Definition 2.1. A sequence $\Theta = (T_n)_{n \in \mathbb{N}}$ is called LR-sequence if

1. Θ is uniformly bounded;
2. $\lim_{n \rightarrow \infty} \|T_n \circ (T_m - I)x\| = 0$ for every m and for every $x \in X$;
3. $\lim_{n \rightarrow \infty} \|(T_m - I) \circ T_n x\| = 0$ for every m and for every $x \in X$.

Many examples of LR-sequences appear in the investigation of operator semi-groups. The Cesaro averages of a power bounded operator form an LR-sequence and moreover encompass Cesaro averages of higher orders. Also for the conditional expectation C_n , $(I - C_n)_{n \in \mathbb{N}}$ forms an LR-sequence. For more details, we refer to [4] -[7].

The following theorem is the main analytic tool in the investigation of LR-sequences. For a proof, we refer to [6].

Theorem 2.2. Let Θ be an LR-sequence on a Banach space X . Then the following conditions are equivalent:

- i. The sequence Θ is strongly convergent.
- ii. $X = \text{Fix}(\Theta) \oplus \overline{\cup_{n \in \mathbb{N}} (I - T_n)X}$.
- iii. The sequence $(T_n x)_{n \in \mathbb{N}}$ has a weak cluster point for every $x \in X$.
- iv. The fixed space $\text{Fix}(\Theta)$ separates the fixed space $\text{Fix}(\Theta')$ of the adjoint operator sequence $\Theta' = (T'_n)_{n \in \mathbb{N}}$ in X' .

If one of the above conditions holds then the strong limit of Θ is a projection onto $\text{Fix}(\Theta)$.

3. CONSTRICTORS

The constrictiveness of an operator is introduced in order to characterize asymptotically periodic Markov operator on L^1 -spaces. Many authors have extended this notion to more general situations. All these notions have in common the general principal reflected by the notion of attractor introduced in [11].

Definition 3.1. Let $\Theta = (T_n)_{n \in \mathbb{N}}$ be an operator sequence on a Banach space X and $C \subseteq X$. Then C is called a constrictor of Θ if

$$\lim_{n \rightarrow \infty} \text{dist}(T_n x, C) = 0$$

for all $x \in B_X := \{z \in X : \|z\| \leq 1\}$.

Our aim is to find conditions on the constrictor C implying nice asymptotic properties of Θ . The first property is that every LR-sequence possessing a weakly compact constrictor is strongly convergent [6]. Moreover, in L^1 -spaces if $C = W + \eta B_E$ where W is weakly compact and $0 \leq \eta < 1$, then an LR-sequence is strongly convergent. Emel'yanov proved also that every LR-sequence containing a weakly compact operator is strongly convergent [7].

4. ERGODICITY OF LR-SEQUENCES ON BANACH LATTICES

For our notation and terminology, we refer to [1, 15, 18]. Consider, the order ideal $E_e := \cup\{[-ne, ne] : n \geq 0\}$ for any $e \in E_+$. If E_e is norm-dense in Banach lattice E then $e \in E_+$ is called a quasi interior point of E_+ . Moreover, let Θ be a positive operator sequence on E , then $x \in E$ is called a positive fixed vector of maximal support if $x \in \text{Fix}(\Theta) \cap E_+$ and every $y \in \text{Fix}(\Theta) \cap E_+$ is contained in the band generated by x . For every quasi-constrictive Markov operator there exists an invariant density with maximal support, see [13]. For positive contraction operator T on KB-spaces with quasi-interior point, being $C := [-z, z] + \eta B_E$ constrictor of T where $z \in E_+$ and $0 \leq \eta < 1$ is proven by R biger, [16]. Then either T is mean ergodic or there is a positive fixed vector $y \neq 0$ of T of maximal support and for such positive fixed vector of maximal support $((I - P_y)A_n^T)_n$ converges strongly to zero where P_y is the band projection from E onto the band generated by y and A_n^T is the Cesaro averages of T .

The main theorem in [9] is firstly generalized on L^1 -spaces for Markov LR-sequences in [5]. The principal tool in the proof of the main results of [5] was using the additivity of the norm on the positive part of the L^1 -space. Since this is no longer the case for a general KB-space, we use different ideas in this paper, inspired by [16].

Theorem 4.1. *Let E be a KB-space with quasi-interior point e and $\Theta = (T_n)_{n \in \mathbb{N}}$ be a positive LR-sequence in E , W be a weakly compact subset of E , and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ such that*

$$\lim_{n \rightarrow \infty} \text{dist}(T_n x, W + \eta B_E) = 0$$

for any $x \in B_E := \{z \in E : \|z\| \leq 1\}$. Then Θ converges strongly.

Proof: The proof is motivated by the proof of Theorem 5.3 in R biger's paper [16]. In the first case, $(T'_n \phi)_{n \in \mathbb{N}}$ is a weak*-nullsequence for each $\phi \in E'$. Then $(T_n x)_{n \in \mathbb{N}}$ has a zero as a weak cluster point for each $x \in E$ and hence by Theorem 2.2 our LR-sequence converges strongly to zero.

In the second case, there is $\phi \in E'_+$ such that $(T'_n \phi)_{n \in \mathbb{N}}$ is not $\sigma(E', E)$ -convergent to 0. Let $0 \neq \psi \in E'_+$ be a $\sigma(E', E)$ -cluster point of $(T'_n \phi)$. We may assume $\|\psi\| = 1$. Then for all $\epsilon > 0$, there exists n_ϵ such that $|\langle \psi, x \rangle - \langle T'_n \phi, x \rangle| < \epsilon$ and $|\langle T'_m \psi, x \rangle - \langle T'_m T'_n \phi, x \rangle| < \epsilon$ for every $m \in \mathbb{N}$ and for every $n \geq n_\epsilon$. Therefore we get $T'_m \psi = \psi$.

Now for fix $\epsilon > 0$ satisfying $\epsilon \leq 1 - \eta$ choose $x \in B_E \cap E_+$ such that $\langle \psi, x \rangle > 1 - \epsilon$. Let $x'' \in E''_+$ be a weak-cluster point of $(T_n x)$. Then there exists n_ϵ such that $\langle \psi, x'' \rangle - \langle \psi, T_n x \rangle < \epsilon$ and $\langle T'_m \psi, x'' \rangle - \langle T'_m \psi, T_n x \rangle < \epsilon$ by combining these two estimates with the property of LR-sequence, we obtain $T''_m x'' = x''$ for every $m \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \text{dist}(T_n x, W + \eta B_E) = 0$, W is weakly compact and E is a band in E'' means E is weak*-closed in E'' then we obtain $x'' \in W + \eta B_E$. Moreover x'' is a weak*-cluster point of $(T_n x)$, then for every $\epsilon' > 0$, there exists n_ϵ such that

$|\langle \psi, x'' \rangle - \langle \psi, T_{n_\epsilon} x \rangle| < \epsilon'$. Therefore we have $|\langle \psi, x'' \rangle - \langle T'_{n_\epsilon} \psi, x \rangle| < \epsilon'$ and since $T'_n \psi = \psi$, we obtain $|\langle \psi, x'' \rangle - \langle \psi, x \rangle| < \epsilon'$. By arbitrariness of ϵ' , $\langle \psi, x'' \rangle = \langle \psi, x \rangle$.

Being E a KB-space, by Theorem 1.1 E is a band in E'' . Denote by P the band projection from E'' onto E i.e. $P : E'' \rightarrow E$. Hence

$$\begin{aligned} (4.1) \quad \langle \psi, Px'' \rangle &= \langle \psi, x'' \rangle - \langle \psi, (I_{E''} - P)x'' \rangle \\ &= \langle \psi, x \rangle - \langle \psi, (I_{E''} - P)x'' \rangle \\ &> 1 - \epsilon - \eta > 0 \end{aligned}$$

It follows from (4.1) that $Px'' \neq 0$. Since x'' is a weak*-cluster point of $(T_n x)_{n \in \mathbb{N}}$, $Px'' > 0$ and moreover, since E has order continuous norm $z := \lim T_n Px'' \in E_+$ exists. Clearly $T_n z = z$ and from $\langle \psi, z \rangle = \langle \psi, Px'' \rangle > 0$, it follows that $z \neq 0$. Hence $\text{Fix}(\Theta) \cap E_+ \neq \{0\}$.

The existence of a quasi-interior point e implies the existence of a strictly positive linear functional ψ [[1], Theorem 12.14]. For $x \in E$, let P_x be the band projection from E onto the band generated by x . Consider the element $\alpha := \sup_{x \in \text{Fix}(\Theta) \cap E_+} \langle \psi, P_x e \rangle > 0$. Choose a sequence $x_n \in \text{Fix}(\Theta) \cap E_+$, $n \in \mathbb{N}$, $\|x_n\| \leq 1$

and $\alpha = \lim \langle \psi, P_{x_n} e \rangle$. Define $u = \sum_n 2^{-n} x_{\lambda_n}$, then u is also an element of $\text{Fix}(\Theta) \cap E_+$ and in addition $P_u \geq P_{x_n}$ for all $n \in \mathbb{N}$. Hence $\langle \psi, P_u e \rangle = \alpha$.

Further taking $x \in \text{Fix}(\Theta) \cap E_+$, clearly $P_{u+x} \geq P_x$ and $P_{u+x} \geq P_u$. From the limit property of $\langle \psi, P_{x_n} \rangle$, $\alpha \leq \langle \psi, P_{u+x} e \rangle \leq \alpha$, so $\alpha = \langle \psi, P_{u+x} e \rangle$ and we know above ψ is strictly positive, it implies that $P_u e = P_{u+x} e$. Owing to quasi-interior point e , $P_u = P_{u+x}$ and by $P_{u+x} \geq P_x$ then we obtain $P_u \geq P_x$. Hence u has a maximal support.

In the next step, we will prove that for the band projection denoted by P_u , of positive fixed vector of maximal support, $((I - P_u)T_n)$ converges to zero strongly as $n \rightarrow \infty$. Let P_u be a band projection onto B_u where $B_u = \overline{\cup_n [-nu, nu]}$. Denote the new operator $Q = I_E - P_u$ and the sequence $\mathcal{S} = (S_n) = (QT_n)$. Since u is a fixed vector so $T_n B_u \subset B_u$ and hence we get $T_n P_u = P_u T_n P_u$ and in addition $QT_n Q = QT_n$ for each n .

Our aim is to show that $(QT_n) = (S_n)$ converges strongly to zero. If not, then there exists by above in the second case of proof, $0 \neq \psi \in \text{Fix} \mathcal{S}' \cap E'_+$.

$$\psi = S'_n \psi = T'_n Q' \psi = Q' T'_n Q' \psi = Q' S'_n \psi = Q' \psi$$

and hence $\psi = S'_n \psi = T'_n Q' \psi = T'_n \psi$, namely, $\psi \in \text{Fix}(\Theta')$. By above, there is $0 \neq x \in \text{Fix}(\Theta) \cap E_+$ such that $\langle x, \psi \rangle > 0$. Then

$$0 < \langle x, \psi \rangle = \langle x, Q' \psi \rangle = \langle Qx, \psi \rangle$$

implies $Qx \neq 0$, i.e. $x \notin B_u$. It is a contradiction to our assumption on u . Indeed, $S_n \rightarrow 0$ strongly as $n \rightarrow \infty$.

In the next step, we will prove that (T_n) converges strongly by using Theorem 2.2.

We know that our operator sequence is an LR-sequence and positive. For fixed $\epsilon > 0$ and $x \in E$, $\lim_{n \rightarrow \infty} S_n x = \lim_{n \rightarrow \infty} (I - P_u)T_n x = 0$, there exists n_ϵ such that $\text{dist}(T_{n_\epsilon} x, B_u) \leq \frac{\epsilon}{3M}$ where $M = \sup_n \|T_n\|$. It implies that there exists $c_\epsilon \in \mathbb{R}_+$ and $y \in [-c_\epsilon u, c_\epsilon u]$ satisfying $\|T_{n_\epsilon} x - y\| \leq \frac{\epsilon}{2M}$.

For any $m \in \mathbb{N}$, $\|T_m T_{n_\epsilon} x - T_m y\| \leq \|T_m\| \|T_{n_\epsilon} x - y\| \leq \frac{\epsilon}{2}$. Moreover, since $[-u, u]$ is Θ -invariant then we get $T_n y \in [-c_\epsilon u, c_\epsilon u]$ for each n . Therefore

$$\text{dist}(T_{n_\epsilon} x, [-c_\epsilon u, c_\epsilon u]) \leq \epsilon'$$

, i.e. for any $\epsilon' > 0$, there exists an interval $[-c_\epsilon, c_\epsilon]$ such that $(T_n x)_{n=0}^\infty \subseteq [-c_\epsilon, c_\epsilon] + \epsilon' B_E$. It shows that $(T_n x)$ has a weak cluster point because E is a KB-space and almost order bounded subset of E is weakly precompact. Then by Theorem 2.2, $(T_n x)_{n \in \mathbb{N}}$ is norm convergent for any $x \in E$, i.e. Θ converges strongly. \square

The previous theorem can be formulated for the net case as follows:

Theorem 4.2. *Let E be a KB-space with quasi-interior point e and $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ be a positive LR-net in E which has a cofinal subsequence, W be a weakly compact subset of E , and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ such that*

$$\lim_{\lambda \rightarrow \infty} \text{dist}(T_\lambda x, W + \eta B_E) = 0$$

for any $x \in B_E := \{z \in E : \|z\| \leq 1\}$. Then Θ converges strongly.

The theorem is also true if we replace a weakly compact subset W of E by an order interval $[-g, g]$ for any $g \in E_+$ because in KB-spaces, every order intervals are weakly compact. Besides in this case we have more results that also dimension of fixed space is finite.

Theorem 4.3. *Let E be a KB-space with a quasi-interior point e , $\Theta = (T_n)_{n \in \mathbb{N}}$ be a positive LR-sequence. Then the following are equivalent*

i *there exists a function $g \in E_+$ and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ such that*

$$\lim_{n \rightarrow \infty} \text{dist}(T_n x, [-g, g] + \eta B_E) = 0, \quad \forall x \in B_E$$

ii *the sequence Θ is strongly convergent and $\dim \text{Fix}(\Theta) < \infty$.*

Proof: The proof of this theorem is motivated by the proof of Theorem 3 in [5].

(i) \Rightarrow (ii) : By the previous theorem, Theorem 4.1, Θ converges strongly onto $\text{Fix}(\Theta)$. Therefore for each $x \in B_E$ $Px \in [-g, g] + \eta B_E$ by the statement of the theorem. If we consider the iteration of Px then we obtain

$$Px = P^2 x \in [-Pg, Pg] + \eta P(B_E).$$

Since P is a projection, i.e. $\|P\| \leq 1$,

$$Px = P^2 x \in [-Pg, Pg] + [-\eta g, \eta g] + \eta^2 B_E.$$

If we repeat of the iteration, we have for arbitrary $n \in \mathbb{N}$,

$$Px = P^n x \in \left[-\sum_{i=0}^{n-1} \eta^i P^{n-i} g, \sum_{i=0}^{n-1} \eta^i P^{n-i} g\right] + [-\eta^{n-1} g, \eta^{n-1} g] + \eta^n B_E.$$

Hence

$$Px \in \left[-\sum_{i=0}^{n-1} \eta^i Pg, \sum_{i=0}^{n-1} \eta^i Pg\right] + [-\eta^{n-1} g, \eta^{n-1} g] + \eta^n B_E.$$

If we continue to iterate Px we get the condition that $Px \in [-c, c]$ where $c = \frac{1}{1-\eta}Pg$. Hence $P(B_E) \subseteq [-c, c]$ that is to say $\text{Fix}(\Theta)$ is finite dimensional space, [18].

(ii) \Rightarrow (i) : If $\dim(\text{Fix}(\Theta)) < \infty$, then there exists a family of pairwise disjoint densities u_1, u_2, \dots, u_k such that $\text{Fix}(\Theta) = \text{span}\{u_1, u_2, \dots, u_k\}$. Denote the element $g := u_1 + \dots + u_k$ and taking an element from $B_E \cap E_+$, then $Px := \lim T_n x$ is a linear combination of u_1, \dots, u_k say $Px = \sum_{i=1}^k \alpha_i u_i \leq \sum_{i=1}^k u_i$. Thus

$$\limsup_{n \rightarrow \infty} \|(T_n x - g)_+\| = \|(Px - g)_+\| = 0$$

for every $x \in B_E \cap E_+$.

□

In the above two theorems, KB-space conditions cannot be omitted. Even for Banach lattices with order continuous norm, this result can fail, for the counterexample, see [8].

Additionally we also can be formulated the above theorem for the net case as follows:

Theorem 4.4. *Let E be a KB-space with quasi-interior point e and $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ be a positive LR-net in E which has a cofinal subsequence. Then the following are equivalent*

- i *there exists a function $g \in E_+$ and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ such that*

$$\lim_{\lambda \rightarrow \infty} \text{dist}(T_\lambda x, [-g, g] + \eta B_E) = 0, \quad \forall x \in B_E$$

- ii *the sequence Θ is strongly convergent and $\dim \text{Fix}(\Theta) < \infty$.*

5. ASYMPTOTIC STABILITY OF LR-SEQUENCES

The asymptotic stability of positive operators and lower-bound technique is developed in applications of Markov operators. In this section we prove the following theorems as a corollary of Theorem 4.3. Theorem 5.5 is the generalization of Theorem 4 in [5]. Emelyanov and Erkursun proved asymptotic stability and existence of lower bound function are equivalent for Markov LR-nets on L^1 -spaces. In this section we will have a KB-space as well.

In the first, we give the following two definitions which are motivated for operator nets by the definitions used in [12].

Definition 5.1. Let Θ be an operator nets on KB-spaces. Θ is called asymptotically stable whenever there exists an element $u \in E_+ \cap U_E$ where $U_E := \{x \in E : \|x\| = 1\}$ such that

$$\lim_{\lambda \rightarrow \infty} \|T_\lambda x - u\| = 0$$

for every element from $E_+ \cap U_E$.

Definition 5.2. An element $h \in E_+$ is called lower bound element for Θ if

$$\lim_{\lambda \rightarrow \infty} \|(h - T_\lambda x)_+\| = 0$$

for every element $x \in E_+ \cap U_E$

For main results, positivity is not only sufficient in addition we need Markov operators. Before proving the theorem, we need to define Markov operator nets on a Banach lattice E .

Definition 5.3. Let E be a Banach lattice. A positive, linear, uniform bounded operator net $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ is called a Markov operator net if there exists a strictly positive element $0 < e' \in E'_+$ such that $T'_\lambda e' = e'$ for each $\lambda \in \Lambda$.

If we consider in the sequence case $\Theta = (T_n)_{n \in \mathbb{N}}$, each element of Θ is Markov operator, still we need a common fixed point e' . For instance even on L_1 -space, the element T_m of Θ might not be a Markov operator on the new norm space (L_1, e'_n) for each $n \neq m \in \mathbb{N}$.

As a remark if we consider the net as $\Theta = (T^n)_{n \in \mathbb{N}}$ as the iteration of a single operator T , Markov operator sequence Θ means T is power-bounded. It is the general version of the definition in [10] where T is contraction. It is well known that if T is a positive linear operator defined on a Banach lattice E , then T is continuous. It is also well known that if the Banach lattice E has order continuous norm, then the positive operator T is also order continuous. We note that the Markov operators, according to this definition, are again contained in the class of all positive power-bounded and that the adjoint T' is also a positive and power-bounded. For more details, we refer to [10].

In the following theorem, we will establish the asymptotic properties of Markov LR-sequences. It is firstly given on L^1 -spaces in [5] which are the examples of Markov LR-sequences which need not to be \mathcal{T} -ergodic sequences. Now we will prove this results on KB-spaces. Before them, we need technical tools for proof. The technical lemma connects norm convergence of order bounded sequences in KB-spaces with convergence in (E, e') for suitable linear forms $e' \in E'$. Recall that $e' \in E'$ is strictly positive if $\langle x, e' \rangle > 0$ for all $x \in E_+ \setminus \{0\}$. We refer to [16] for proof of the lemma.

Lemma 5.4. *Let $(x_n)_{n \in \mathbb{N}}$ be an order bounded sequence in a KB-space and let $x' \in E'$ be strictly positive. Then $\lim_{n \rightarrow \infty} \|x_n\| = 0$ if and only if $\lim_{n \rightarrow \infty} \langle |x_n|, x' \rangle = 0$.*

Theorem 5.5. *Let Θ be a Markov LR-sequence on KB-spaces with fixed common element e' . Then the following are equivalent:*

- i Θ is asymptotically stable
- ii Θ has nontrivial lower-bound element in the space (E, e')
- iii Θ has nontrivial lower-bound element in the space (E, e') for each $e' \in E'_+$.

Proof: (i) \Rightarrow (iii) : Let $g \in E_+ \cap U_E$ satisfy

$$\lim_{t \rightarrow \infty} \|A_t^T x - g\| = 0$$

for every $x \in E_+ \cap U_E$, then g is automatically a nontrivial lower-bound function for Θ on the space (E, e') for each strictly positive $e' \in E'_+$.

(iii) \Rightarrow (ii) : Obvious

(ii) \Rightarrow (i) : Let h be a lower-bound element of Θ in (E, e') , i.e.,

$$\lim_{n \rightarrow \infty} \langle (T_n x - h)_+, e' \rangle = 0 \quad \forall x \in E_+ \cap U_E.$$

Since the norm on (E, e') is an L_1 -norm, then we can consider

$$\limsup_{n \rightarrow \infty} \langle (T_n x - h)_+, e' \rangle \leq \eta$$

where $\eta := 1 - \|h\|_{(E, e')} = 1 - \langle h, e' \rangle$ for each $x \in E_+ \cap U_E$.

By Theorem 4.3, \tilde{T}_n where $\tilde{T}j_{e'}x = j_{e'}Tx$ for lattice homomorphism $j_{e'} : E \rightarrow (E, e')$ converges strongly to the finite dimensional fixed space of $\tilde{\Theta}$. Therefore by Eberlein's Theorem

$$(E, e') = (\text{Fix}(\tilde{\Theta})) \oplus \text{Ker}(\tilde{\Theta}).$$

In addition we know that $\text{Fix}(\tilde{\Theta})$ is a sublattice of (E, e') and by Judin's Theorem, it possesses a linear basis $(\tilde{u}_i)_{i=1}^n$ where $n = \dim(\text{Fix}(\tilde{\Theta}))$ which consists of pairwise disjoint element with $\|\tilde{u}_i\| = 1$, $i = 1, \dots, n$, see [18]. Since $Tu_i = u_i$ for each $i = 1, \dots, n$,

$$\langle (h - u_i)_+, e' \rangle = \langle (h - Tu_i)_+, e' \rangle = \lim_{t \rightarrow \infty} \langle (h - T_n u_i)_+, e' \rangle = 0$$

implies

$$(5.1) \quad u_i \geq h \geq 0 \quad i = 1, \dots, n.$$

Since $(\tilde{u}_i)_{i=1}^n$ is pairwise disjoint with $\|\tilde{u}_i\|_{(E, e')} = 1$ the condition 5.1 ensure that $\dim(\text{Fix}(\tilde{\Theta})) = 1$. Therefore $\text{Fix}(\tilde{\Theta}) = \mathbb{R}\tilde{u}_1$ and for every element $x \in E_+ \cap B_E$, $\lim_{n \rightarrow \infty} T_n x = u_1$. □

The Lasota's criterion of asymptotic stability says that a one-parameter Markov semigroup if and only if there is a nontrivial lower-bound function. In [5] Lasota's lower-bound criteria is generalized on L^1 -spaces to abelian Markov semigroups. In this proposition we generalize it on KB-spaces. An abelian Markov semigroup is an operator net with respect to the natural partial order \succ mentioned Section 2.

Proposition 5.6. Every asymptotically stable abelian Markov semigroup $\mathcal{T} = (T_t)_t$ which has a common fixed point e' of \mathcal{T}' on KB-space is an LR-net.

Proof: Since any Markov operator on KB-space is a positive contraction, a Markov sequence is uniformly bounded. We need to check (LR2) and (LR3) conditions of Definition 2.1. Moreover because of abelian property, it suffices to prove only (LR2) or (LR3). Without loss of generality, taken for an arbitrary element x from $E_+ \cap U_E$. Since \mathcal{T} is asymptotically stable then by Lemma 5.4 $\lim_{n \rightarrow \infty} \langle |T_{t_n} x - u|, e' \rangle = 0$. If we consider fixed $t_m \in \mathbb{R}$, $\langle ((I - T_{t_m})x)_+, e' \rangle = \langle ((I - T_{t_m})x)_-, e' \rangle$ by additivity property of (E, e') .

Now for (LR2), call $(I - T_{t_m})x = y$

$$\begin{aligned} \langle |T_{t_n}(I - T_{t_m})x|, e' \rangle &= \langle |T_{t_n}y|, e' \rangle = \langle |T_{t_n}y_+ - T_{t_n}y_-|, e' \rangle \\ &= \left\langle |T_{t_n}y_+ - \|y_+\|_{(E, e')} u + \|y_-\|_{(E, e')} u - T_{t_n}y_-|, e' \right\rangle \\ &\leq \left\langle |T_{t_n}y_+ - \|y_+\|_{(E, e')} u|, e' \right\rangle \\ &\quad + \left\langle |T_{t_n}y_- - \|y_-\|_{(E, e')} u|, e' \right\rangle \end{aligned}$$

which converge to zero as $n \rightarrow \infty$. Therefore $\lim_{n \rightarrow \infty} T_{t_n}(I - T_{t_m})x = 0$ for each $x \in U_E \cap E_+$. □

Proposition 5.7. Let $\mathcal{T} = (T_t)_t$ be an abelian Markov semigroup which has a common fixed point e' of \mathcal{T}' on KB-spaces possessing a nontrivial lower-bound function on the space (E, e') , then \mathcal{T} is an LR-net.

Proof: The important part of the proof is that since $T_{t'}$ is Markov then $\langle ((I - T_{t'})f)_+, e' \rangle = \langle ((I - T_{t'})f)_-, e' \rangle$ for each t' and for each $f \in E$ on the space (E, e') . It implies that $\|((I - T_{t'})f)_+\|_{(E, e')} = \|((I - T_{t'})f)_-\|_{(E, e')}$. We repeat the argument in [5] in short for convenient of the reader.

Let $0 \neq h \in E_+$ be a nontrivial lower-bound element for \mathcal{T} on the space (E, e') , then $\|h\|_{(E, e')} \leq \|(h - T_t f)_+\|_{(E, e')} + \|h \wedge T_t f\|_{(E, e')} \leq \epsilon + 1$ for every t so obviously $\|h\|_{(E, e')} \leq 1$. Since semigroup is Markovian then it is uniformly bounded. Moreover because of abelian property, it suffices to prove only (LR2) or (LR3). Thus we prove the following formula.

$$(5.2) \quad \lim_{t \rightarrow \infty} \|T_t(I - T_{t'})f\|_{(E, e')} = 0 \quad (\forall t', f \in E)$$

Take any element $f \in B_E$, then we know that $T_{t'}$ is Markov and then

$$(5.3) \quad \langle (I - T_{t'})f_+, e' \rangle = \langle (I - T_{t'})f_-, e' \rangle$$

Therefore by 5.2, we have to prove that $\lim_{t \rightarrow \infty} \|T_t f\|_{(E, e')} = 0$ for every $f \in B_E$ such that 5.3 holds. Define the set $E_0 := \{f \in E : \|f_+\|_{(E, e')} = \|f_-\|_{(E, e')}\}$.

Take any element $f \in E_0$ such that $f = 2^{-1} \|f\|_{(E, e')} (f_1 - f_2)$ where $f_1 = 2 \|f\|_{(E, e')}^{-1} f_+$ and $f_2 = 2 \|f\|_{(E, e')}^{-1} f_-$. Hence f_1 and f_2 are elements of $E_+ \cap U_E$.

Since h is the lower-bound element for the Markov semigroup \mathcal{T} , there exists t_1 such that $\|(h - T_{t_1} f_1)_+\|_{(E, e')} \leq \frac{1}{4} \|h\|_{(E, e')}$ and $\|(h - T_{t_1} f_2)_+\|_{(E, e')} \leq \frac{1}{4} \|h\|_{(E, e')}$ hold for every $t \geq t_1$. From Riesz space properties, we obtain $\|T_{t_1} f_1 - T_{t_1} f_2\|_{(E, e')} \leq 2 - \frac{1}{2} \|h\|_{(E, e')}$ and $\|T_t f\|_{(E, e')} \leq (1 - \frac{1}{4} \|h\|_{(E, e')}) \|f\|_{(E, e')}$ for every $t \geq t_1$.

Replacing f with $T_{t_1} f$ which is also an element of E_0 and repeating the argument above gives an element t_2 such that

$$\|T_t T_{t_1} f\|_{(E, e')} \leq (1 - \frac{1}{4} \|h\|_{(E, e')}) \|T_{t_1} f\|_{(E, e')} \quad \forall t \geq t_2$$

By induction, we can generate a sequence (t_n) such that

$$\begin{aligned} \|T_t f\|_{(E, e')} &\leq \|T_t T_{t_{n-1}} f\|_{(E, e')} \leq (1 - \frac{1}{4} \|h\|_{(E, e')}) \|T_{t_{n-1}} f\|_{(E, e')} \\ &\vdots \\ (\forall t \geq t_1 + \dots + t_n) &\leq (1 - \frac{1}{4} \|h\|_{(E, e')})^n \|f\|_{(E, e')} \end{aligned}$$

Since $0 < \|h\| < 1$, then $\lim_{t \rightarrow \infty} \|T_t f\|_{(E, e')} = 0$ and hence the proof is completed. \square

Theorem 5.8. *Let $\mathcal{T} = (T_t)_t$ be an abelian Markov semigroup which has a common fixed point e' of \mathcal{T}' on KB-spaces E . Then the following are equivalent:*

- i \mathcal{T} is asymptotically stable
- ii There exists a nontrivial lower-bound element for \mathcal{T} in the space (E, e')

Proof: Since the asymptotic stable Markov semigroup \mathcal{T} is the LR-sequence by 5.6, the existence of nontrivial lower-bound element for \mathcal{T} follows from Theorem 5.5. In addition, the existence of nontrivial lower bound element for \mathcal{T} gives us that \mathcal{T} is an LR-sequence by Proposition 5.7 and the asymptotic stability of \mathcal{T} follows from Theorem 5.5. \square

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